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Thy Consistent Criterion for Stationary Gaussion Statistical Structures

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ABSTRACT

In the paper there are discussed Gaussion statistical structures $\{E, S, \mu_h,\ h \in H\}$ in Hilbert space of measures. We prove necessary and sufficient conditions for existence of such criterion in Hilbert space of measures.

Key words: Consistent criterion, orthogonal, strongly separable statistical structures.
Classification cocles 62H05, 62H12

Let there is given (E,S) measurable space and on this space there given $\{\mu_h,\ h \in H\}$ family of probabilitg measures defined on S, The H set of hypotheses. Thy following definitions are taken from thy works ([1]-[5]).

Definition 1. A statistical structure is is called object $\{E, S, \mu_h,\ h \in H\}$

Definition 2. A statistical structure $\{E, S, \mu_h,\ h \in H\}$ is called orthogonal (singular) (O) if thy family of probability measures $\{\mu_h,\ h \in H\}$ are pairwise singular measures.

For $\{\mu_h,\ h \in H\}$ be probability measures defined on thy measurable space (E,S). For each $h \in H$ denote by $\overline{\mu}_h$ thy completion of thy measure μ_h and denote by dom($\overline{\mu}_h$) thy σ-algebra of all $\overline{\mu}_h$-measurable subsets of E.

Let $S_1 \bigcap_{h \in H} \mathrm{dom}(\overline{\mu}_h)$.

Definition 3. A statistical structure $\{E, S_1, \overline{\mu}_h,\ h \in H\}$ is called strongly separable if there exists thy family of S_1 - measures sets $\{Z_h,\ h \in H\}$ such that the relations are fulfilled:

1) $\overline{\mu}_h(Z_a) = 1, \quad \forall h \in H;$
2) $Z_{h_1} \cap Z_{h_2} = \varnothing \quad \forall h \in H;$
3) $\bigcup_{h \in H} Z_h = E.$

Definition 4. We consider the concept of the hypothesis as any assumption that determines the form of the distribution of the population.

Let H be set of hypotheses and B(H) be σ-algebra of subsets of H which contains all finite subsets of H.

Definition 5. We will say that the statistical structure $\{E, S\ , \mu_h\ ,\ h \in H\}$ admits a consistent criterion (CC) for testing hypothesis if there exists at least one measurable mapping

$$\delta : (E, S) \to (H, B(H)),$$

Such that

$$\mu_h(\{x : \delta(x) = h\}) = 1, \quad \forall h \in H.$$

Remark 1. The notion and corresponding construction of consistent criterion for testing hypotheses was introduced and sudid by Z. Zerakidze (see [5]).

Let M^{σ} be real linear space of all alternating finite measures on S.

Definition 6. A linear subset $M_H \subset M^{\sigma}$ is called a Hilbert space of measures if:

1) On M_H one can introduce the scalar product (μ,ν), $\mu,\nu \in M_H$ with respect to which M_H is the Hilbert space and for all mutually singular measures μ and ν, $\mu,\nu \in M_H$, the scalar product $(\mu,\nu)=0$;

2) If $\nu \in M_H$ and $|f(x)| \le 1$ then $\nu_f(A) = \int_A f(x)\nu(dx) \in M_H$, where f(x) is S-measurable real function and $(\nu_f,\nu_f) \le (\nu,\nu)$;

If $\nu_n \in M_H$, $\nu_n > 0$, $\nu_n(E) < \infty$, n=1,2,…. And $\nu_n \downarrow 0$ then for any $\mu \in M_H$ $\lim_{n\to\infty}(\nu_n \mu) = 0$.

Remark 2. The notion and corresponding construction of consistent criterion of the Hilbert space of measures was introduced and sudid by Z.Zerakidze (see [4]).

Let $\xi_h(t,\omega) = \theta_h(t) + \Delta(t,\omega), \quad t \in T \subset R, \ \forall h \in H$

Gaussian real processes, where T be closed bounded subset of R, with zero means $E(\Delta(t,\omega)) = 0, \ \ E\xi_h(t,\omega) = \theta_h(t), \quad t \in T$ and correlation function

$E(\Delta(t,\omega)\Delta(k,\omega)) = E\xi_h(t,\omega)\xi_h(k,\omega) = R(t-k)$

Card H=continuum. Let $\mu_{\Theta_h}, h \in H$, card (H)=c be the corresponding probability measures given on S and $f_h(\lambda), \lambda \in R, \forall h \in H$ spectral measures densities such that relations are fulfilled: $(1+\lambda^2)^{-N} K_h \le f_h(\lambda) \le C_h (1+\lambda^2)^{-N}, \quad h \in H,$ where K_h and $C_h, h \in H$ are positive constants. We shall assume that the functions itself or its derivatives satisfies conditions: $\int_{-\infty}^{+\infty} [\Theta_h^{(m)}(t)]dt = \infty \ \ \forall h \in H$, m=0,1,2,…..n.

Then the corresponding probability measures μ_{h_1} and μ_{h_2} are pairwise orthogonal $\forall h_1, h_2, \ \ \forall h_1 \ne h_2 \in H$ (see[1]) and $\{E, S_1, \overline{\mu}_{\theta_h}, \ h \in H\}$, CardH=C are Gaussian orthogonal ststionary statistical structures. Next we consider S-measurable $g_h(x), \ h \in H$ functions, such that $\sum_{h \in I_h} \int_E |g_h(x)|^2 \ \mu_{\theta_h}(dx) < \infty$ where $I_h \in H$ a countable subsets in H. Let M_h the set measures defined by formula $\nu(B) = \sum_{h \in I_h} \int_B g_h(x)\mu_{\theta_h}(dx)$, define a scalar product by formula

$(\nu_1,\nu_2) = \sum_{h \in I_{h_1} \cap I_{h_2}} \int_B g_h^1(x) g_h^2(x) \mu_{\theta_h}(dx)$ where $I_{h_1} \subset H$, $I_{h_2} \subset H$ a countable subsects in H.

1. We will show M_H is Hilbert space.

Let $\psi_n(B) = \sum_{h \in I_h} \int_B g_h(x)\mu_{\theta_h}(dx)$

Here $I_{h_n} \subset H$, n=1,2,….a couutable subsets is H and ψ_n is fundament sequence in M_H. Let

$I' \subset \bigcup_{n=1}^{\infty} I_{h_n}, \quad CardI' < c$

So the Gaussian orthogonal statistical structure $\{E, S, \mu_{\theta_h}, \ h \in H\}$ is strongly separable statistical structure the instead of this functional $g_h^n(x) I_{C_h}(x) \ (C_h \cap C_{h'} = \varnothing, h \ne h')$ then

$\psi_n(B) = \sum_{h \in I'} \int_{B \cap C_h} g_h^n(x)\mu_{\theta_h}(dx), \quad \forall n \in N,$

Let $g^n_{I'}(x)=\sum_{h\in I'} g^n_h(x)$,

It is clear, that

$\|\psi_n-\psi_m\|^2=\int |g^n_{I'}(x)-g^m_{I'}(x)|\mu_{I'}(dx)$.

As will as $L^2(\mu_{I'})$ space is complete space, then exists such function $g^n_{I'}(x)$ thet

$\int g^2_{I'}(x)\mu_{I'}(dx)<\infty \quad \int |g^n_{I'}(x)-g^m_{I'}(x)|\mu_{I'}(dx)\to\infty,\ n\to\infty.$

Let $\psi_n(B)=\sum_{h\in I'}\int_B g_{I'}(x)I_{C_i}(x)\mu_{\theta_h}(dx),\ \|\psi_n - \psi\|,\to\infty, n\to\infty.$

2. If $\nu(B)=\sum_{h\in I'}\int_B g_h(x)\mu_{\theta_h}(dx)$, then $\nu_f(B)=\int_B f(x)\nu(dx)=\sum_{h\in I_0}\int_B f(x)g_h(x)\mu_{\theta_h}(dx),\ I_o\subset H$ and so $|f(x)|\le 1$, then $(\nu_f,\nu_f)=\sum_{h\in I_0}\int |f(x)g_h(x)|^2\mu_{\theta_h}(dx),\ \le \sum_{h\in I_0}\int |g_h(x)|^2\mu_{\theta_h}(dx)=(\nu,\nu)$.

3. Let $\nu=\sum_{h\in I_1}\int g_h(x)\mu_{\theta_h}(dx),\ \mu=\sum_{h\in I_2}\int f_i(x)\mu_{\theta_h}(dx),\ I_1,I_2\subset H$ and $\mu\perp\nu$.

Let $I_3=I_1\bigcup I_2$ and $\mu_{\theta_{h_i}}(C_{h_j})=\begin{cases}1, & \text{if} \quad i=j\\ 0, & \text{if} \quad i\ne j\end{cases}$, $i,j\in I_3,\ C_{h_i}\bigcap C_{h_j}=\varnothing \quad i\ne j$

As $\nu\perp\mu\Rightarrow\sum_{h\in I_3} g_h(x)f_h(x)=0$ almost everywhere with respect μ_{I_3} and $(\nu,\mu)=\int\sum_{h\in I_3} g_h(x)f_h(x)\mu_{\theta_h}(dx)=0$

4. Let $\nu_n\in H_H$, $\nu_n\ge 0,\ \nu_n\downarrow 0,\ \nu_n(E)<\omega,$ then

4 $\nu_n(B)=\sum_{h\in I_n}\int_B g^{(n)}_h(x)\mu_{\theta_h}(dx)\in M_H \quad \forall n\in N$ can be considered $g^n_h\downarrow 0$ and $\nu_n(B)=\sum_{h\in I'}\int_B g^{(n)}_h(x)\mu_{\theta_h}(dx),(\nu_n,\nu_n)=\int\sum_{h\in I'}|g^{(n)}_h(x)|^2 I_{C_h(x)}\mu_{\theta_h}(dx)$ and $(\nu_n,\nu_n)\to o$.

We will show that M_H is Hilbert space of measures.

We denote by F=F(M_H) the set of real functions f such $\int f(x)\overline{\mu}_{\theta_h}(dx)$ is defined $\forall\overline{\mu}_{\theta_h}\in M_H$.

Let $M_H=\oplus_{h\in H} H_2(\overline{\mu}_h)$ be the Hilbert space of measures S_1 $S_1=\bigcap_{h\in H}\mathrm{dom}(\overline{\mu}_{\theta_h})$ E is the complete separable metric space and the Borel σ-algebra in E and card$H\le C$.

Thenthe following theorem holds:

Theorem. In order that the orthogonal stationary Gaussian Statistical structure $\{E,S_1,\overline{\mu}_{\theta_h},\ h\in H\}$ card $H\le C$ admits a consistent criterion for testing hypothesis in the theory (ZFC) ξ (MA) it is necessary and sufficient that the correspondence $f\leftrightarrow\psi_f(f\in F(M_H))$, given by the formula

$\int_E f(x)\overline{\mu}_{\theta_h}(dx)=(\psi_f,\overline{\mu}_{\theta_h}),\ \forall\overline{\mu}_{\theta_h}\in M_H$ was be one-to-one.

Prof. Necessity. The existence of a consistent criterion for testing hypotheses $\delta:(E,S_1)\to(H,B(H))$: Implies that $\overline{\mu}_{\theta_h}(\{x:\delta(x)=h\})=1,\ \forall h\in H.$ Setting $X_h=\{x:\delta(x)=h$ for $\forall h\in H.$ we get:

1) $\overline{\mu}_{\theta_h}(X_h)=\overline{\mu}_{\theta_h}(\{x:\delta(x)=h\})=1,\ \forall h\in H;$

2) $X_{\theta_{h_1}}\bigcap X_{\theta_{h_2}}=\varnothing,\ \forall\, h_1\ne h_2,\ h_1,h_2\in H;$

3) $\bigcup_{h\in H} X_{\theta_h}\equiv\{x:\delta(x)\in h\}=E,$

Therefore the statistical structure $\{E, S_1, \overline{\mu}_{\theta_h},\ h \in H\}$ is strongly separable, hence, there exists S_1-measurable sets $X_{h,}\ h \in H$ such that

$$\overline{\mu}_{\theta_h}(X_{h'}) = \begin{cases} 1, & \text{if} \quad h = h' \\ 0, & \text{if} \quad h \neq h' \end{cases}$$

Let the function $I_{X_h}(x) \in F$ corresponds to $\overline{\mu}_{\theta_h} \in H_2(\overline{\mu}_{\theta_h})$.

Then $\int I_{X_h}(x)\overline{\mu}_{\theta_h}(dx) = \int I_{X_h}(x) I_{X_h}(x)\overline{\mu}_{\theta_h}(dx) = (\overline{\mu}_{\theta_h}, \overline{\mu}_{\theta_h})$.

Let the function $f_{\psi_1}(x) = f_1(x) I_{x_h}(x)$ corresponds to $\psi_1 \in H_2(\overline{\mu}_{\theta_h})$.

Then for every $\psi_2 \in H_2(\overline{\mu}_{\theta_h})$:

$$\int f_{\psi_1}(x) f_{\psi_2}(x)\overline{\mu}_{\theta_h}(dx) = \int f_1(x) f_2(x) I_{x_h}(x) I_{x_h}(x)\overline{\mu}_{\theta_h}(dx) = \int f_1(x) f_2(x)\overline{\mu}_{\theta_h}(dx) = (\psi_1, \psi_2).$$

Further, let the function $f(x) = \sum_{h \in H} \int g_h(x)\overline{\mu}_{\theta_h}(dx)$. Then fort each $\nu_1 \in M_H$., such that $\nu_1 = \sum_{h \in H_{f_1}} \int g^1_h(x)\overline{\mu}_{\theta_h}(dx)$, we have

$$\int f(x)\nu_1(dx) = \int \sum_{h \in H_f \cap H_{f_1}} g_h(x) g^1_2(x)\overline{\mu}_{\theta_h}(dx) = \sum_{h \in H_f \cap H_{f_1}} \int g_h(x) g^1_2(x)\overline{\mu}_{\theta_h}(dx) = (\nu_1, \nu_2).$$

From this discussion it follows that the above – indicated correspondence puts some function into correspondence puts some function $f \in F(M_B)$. into correspondence to each $\psi_f \in M_H$ if we identify in F (M_H) the functions coinciding with respect to the measure $\overline{\mu}_{\theta_h}$, $h \in H$, then this correspondence will be bijective.

Sufficiency. Let $f \in F(M_H)$ corresponds to $\overline{\mu}_{\theta_h} \in M_H$ for wich $\int f(x)\overline{\mu}_{\theta_h}(dx) = (\overline{\mu}_{\theta_h}, \overline{\mu}_{\theta_h})$, then for every $\overline{\mu}_{\theta_h}, \overline{\mu}_{\theta_h} \in M_H$. $\int f_h(x)\overline{\mu}_{\theta_{h'}}(dx) = (\overline{\mu}_{\theta_h}, \overline{\mu}_{\theta_{h'}}) = \int f_1(x) f_2(x)\overline{\mu}_{\theta_h}(dx) = \int f_h(x) f_2(x)\overline{\mu}_{\theta_h}(dx)$.

So $f_h(x) = f_1(x)$ almost everywhere with respect to the measure $\overline{\mu}_{\theta_h}$ and $f_h > 0$, $\int f^2_h(x)\overline{\mu}_{\theta_h}(dx) < +\infty$. If $\overline{\mu}^*_{\theta_h} = \int f^*_h(x)\overline{\mu}_{\theta_h}(dx)$ then $\int f^*_\mu(x)\overline{\mu}_{\theta_{h'}}(dx) = (\overline{\mu}_{\theta_h}, \overline{\mu}_{\theta_{h'}}) = 0, h \neq h'$. On the other hand $\overline{\mu}_{\theta_h}(E - X_h) = 0$, where $X_h = \{x : f^*_h(x) = h\}$.

Hence it follows that

$$\overline{\mu}_{\theta_h}(X_{h'}) = \begin{cases} 1, & \text{if} \quad h = h' \\ 0, & \text{if} \quad h \neq h' \end{cases}$$

Therefore the statistical structure $\{E, S_1, \overline{\mu}_{\theta_h},\ h \in H\}$ Is weakly separable, we represent $\{\overline{\mu}_{\theta_h},\ h \in H\}$, $\text{card} H \le C$ as an inductive sequence $\{\overline{\mu}_{\theta_h},\ h < H\}$, where W! denotes the first ordinal number of the power of the set H.

Since the statistical structure $\{E, S_1, \overline{\mu}_{\theta_h},\ h \in H\}$ is weakly separable, there exists the family of S_1 -- measurable sets $\{X_{h,} h \in H\}$ such that for all $h \in [0, \omega_1)$ we have:

$$\overline{\mu}_{\theta_h}(X_{h'}) = \begin{cases} 1, & \text{if} \quad h = h' \\ 0, & \text{if} \quad h \neq h' \end{cases}$$

We define W_1 sequence Z_h of parts of the space E such that the following relations hold:

1) Z_h is Borel subset of E for alle $h<w_1$;
2) $Z_h \subset X_h$ for all $h<w_1$;
3) $Z_h \cap Z_{h'} = \varnothing$ for all $h< w_1$, $h'<w_1$; $h \neq h'$;
4) $\overline{\mu}_{\theta_h}(Z_h) = 1$ for all $h<w_1$;

Suppose that $Z_{h_0} = X_{h_0}$. Suppose further that the partial sequence $\{Z_{h'}\}_{h'<h}$ is already defined for $h<w_1$. It is clear that $\mu^*(\bigcup_{h'<h} Z_{h'}) = 0$. Thus there exists a Borel subset Y_n of the space E such that the following relations valid: $\bigcup_{h'<h} Z_{h'} \subset Y_h$ and $\mu^*(Y_h) = 0$

Assuming that $Z_h = X_h - Y_h$, we construct the w_1 sequence $\{Z_h\}_{h<\omega_1}$ of disjunctive measurable subsets of the space E. therefore $\overline{\mu}_{\theta_h}(Z_h) = 1 \ \forall h < \omega_1$ and the statistical structure $\{E, S_1, \overline{\mu}_{\theta_h},\ h \in H\}$, $\mathrm{card} H \leq C$ is strongly separable because there exists a family of elements of the σ - algebra $S_1 = \bigcap_{h \in H} \mathrm{dom}(\overline{\mu}_{\theta_h})$ such that:

1) $\overline{\mu}_{\theta_h}(Z_h) = 1 \ \forall h \in H$
2) $Z_h \cap Z_{h'} = \varnothing \quad \forall h, h', h \neq h' \in H$;
3) $\bigcup_{h \in H} Z_h = E$,

For $x \in E$, we put $\delta(x) = h$, where h is the unique hypothesis from the set H for which $x \in Z_h$. The existence of such a unique hypothesis H can be proved using condition 2), 3).

Now let $Y \in B(H)$. Then $\{x : \delta(x) \in Y\} = \bigcup_{h \in Y} Z_h$.

We mosr show that $\{x : \delta(x) \in Y\} \in \mathrm{dom}(\overline{\mu}_{\theta_{h_0}})$ for each $h_0 \in H$.

If $h_0 \in Y$, Then $\{x : \delta(x) \in Y\} = \bigcup_{h \in Y} Z_h = (Z_{h_0}) \cup \left(\bigcup_{h \in Y - \{h_0\}} Z_h \right)$.

On the one round, from the validity of the conditions 1), 2), 3) it follows that

$$Z_{h_0} \in S_1 = \bigcap_{h \in H} \mathrm{dom}(\overline{\mu}_{\theta_h}) \subseteq \mathrm{dom}(\overline{\mu}_{\theta_{h_0}})$$

θ_h The ofter round, the vaidaty of the condition

$$\bigcup_{h \in Y - \{h_0\}} Z_h \subseteq (E - Z_{h_0})$$

Imlies that

$$\mu_{\theta_{h_0}}\left(\bigcup_{h \in Y - \{h_0\}} Z_h \right) = 0,$$

The last equality yelds that $\bigcup_{h \in Y - \{h_0\}} Z_h \in \mathrm{dom}(\overline{\mu}_{\theta_{h_0}})$,

Sience $\mathrm{dom}(\overline{\mu}_{\theta_{h_0}})$ is a σ - algebra, we deduce that

$$\{x : \delta(x) \in Y\} = (Z_{h_0}) \cup \left(\bigcup_{h \in Y - \{h_0\}} Z_h \right) \in \mathrm{dom}(\overline{\mu}_{\theta_{h_0}})$$

If $h_0 \notin Y$, then $\{x:\delta(x)\in Y\}=\bigcup_{h\in Y} Z_h \subseteq (E-Z_{h_0})$

And we conclude that $\overline{\mu}_{\theta_{h_0}}\{x:\delta(x)\in Y\}=0$

The last relation impies that

$\{x:\delta(x)\in Y\}\in \mathrm{dom}(\overline{\mu}_{\theta_{h_0}})$,

Trus we have show the validaty of the relation

$\{x:\delta(x)\in Y\}\in \mathrm{dom}(\overline{\mu}_{\theta_{h_0}})$

For an arbitrary $h_0 \in H$, Hence

$$\{x:\delta(x)\in Y\}\in \bigcap_{h\in H}\mathrm{dom}\left(\overline{\mu}_{\theta_h}\right)=S_1.$$

We have show that the map $\delta:(E,S_1)\to\left(H,B(H)\right)$ is measurable map and we asception that

$\overline{\mu}_{\theta_{h_0}}\{x:\delta(x)=h\}=\overline{\mu}_{\theta_h}(Z_h)=1$

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თანმიმდევრული კრიტერიუმი გაუსის სტაციონარული სტატისტიკური სტრუქტურებისათვის

ზ. ზერაკიძე, ჯ. ქირია, თ. ქირია, ი. ლოლაძე

რეზიუმე

ნაშრომში განხილულია ჰილბერტის ზომების გაუსის სტატისტიკური სტრუქტურებისთვის. ნაპოვნიაა ჰილბერტის ზომების კრიტერიუმის არსებობისთვის აუცილებელია და საკმარისი პირობები.

Последовательный критерий для Гауссовских стационарных статистическх структур

З.С. Зеракидзе, Дж.К. Кирия, Т.В. Кирия, И.Н. Лоладзе

Резюме

В статье обсуждаются Гауссовскиие статистические структуры в Гильбертовом пространстве мер. Мы доказываем необходимые и достаточные условия существования такого критерия в Гильбертовом пространстве мер.