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On the Growth of Vapour Bubble in Metastable Liquid as Variational Problem

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ABSTRACT

Bubble growing process is considered theoretically when bubble consists saturated vapour of liquid and when we have vapour-gas bubble on the base of generalized Rayleigh-Plesset equation. For this purpose we use variational method with help of which we seek for those cases corresponding Euler-Poisson equations integral curves. Obtained EP- equations allow us to find extremals of our variational task. In conclusion, the Rayleigh-Taylor instability is considered general case of nonspherical perturbances for spherical bubbles and the case of the radial perturbances of bubbles.

Key words: Vapour-gas bubble, metastable liquid, variational task, Euler-Poisson equations, Rayleigh-Plesset equation, the Rayleigh-Taylor instability.

Brief report

1. Here we will proceed from generalized equation of Rayleigh-Plesset equation for description of bubbles dynamics [1]:

$$\frac{p_B(t)-p_\infty(t)}{\rho_L}=R\frac{d^2R}{dt^2}+\frac{3}{2}\left(\frac{dR}{dt}\right)^2+\frac{4\nu_L}{R}\frac{dR}{dt}+\frac{2\sigma}{\rho_L R}. \qquad (1)$$

where R is radius of vapour bubble, t is time, $p_B(t)$ is a pressure of saturated vapour in the bubble, $p_\infty(t)$ is a pressure in the liquid, ρ_L is the saturated liquid density, ν_L is the kinematic viscosity of liquid, σ is the surface tension of the bubble.

Having integral from both sides of the equation (1) over the time,

$$\int_{t_0}^{t_1}\frac{p_B(t)-p_\infty(t)}{\rho_L}dt=\int_{t}^{t_1}\left[R\frac{d^2R}{dt^2}+\frac{3}{2}\left(\frac{dR}{dt}\right)^2+\frac{4\nu_L}{R}\frac{dR}{dt}+\frac{2\sigma}{\rho_L R}\right]dt, \quad (2)$$

and reprisent it in the form of functional

$$U[R(t)]=\int_{t_o}^{t_1}F(t,R,R',R'')dt, \qquad (3)$$

where

$$F=RR''+\frac{3}{2}R'^2+\frac{4\nu_L}{R}R'+\frac{2\sigma}{\rho_L R}, \quad (4)$$

and the touch over R denotes dR/dt.

Let investigate on extremum functional (3) where the function F is supposed to be differentiate two times with respect to the time t at the boundary conditions:

$$R(t_0) = R_0,\ R'(t_0) = R'_0,\ R''(t_0) = R''_0;\ R(t_1) = R_1,\ R'(t_1) = R'_1,\ R''(t_1) = R''_1. \quad (5)$$

By means of the functional variation on the curve realizing extremum we obtain:

$$\delta U = \int_{e_0}^{e_1} \left(F_R - \frac{d}{dt} F_{R'} + \frac{d^2}{dt^2} F_{R''} \right) \delta R dt = 0. \quad (6)$$

On arbitrary choice of δR, because of continuity of the expression in brackets with respect to time on the same curve $R(t)$, we obtain Euler-Poisson's equation

$$F_R - \frac{d}{dt} F_{R'} + \frac{d^2}{dt^2} F_{R''} = 0. \quad (7)$$

After determination all values introducing in equation (7)

$$F_R = R'' - \frac{4\nu}{R^2} R' - \frac{2\sigma}{\rho_L} \frac{1}{R^2},\ F_{R'} = 3R' + \frac{4\nu}{R},\ \frac{d}{dt} F_{R'} = 3R'' - \frac{4\nu}{R^2} R' \quad F_{R''} = R,\ \frac{d^2}{dt^2} F_{R''} = R'', \quad (8)$$

and substitution (8) into equation (7) we have following equation

$$R^2 R'' = -\frac{2\sigma}{\rho_L}. \quad (9)$$

After comparing equation (1) and (9) we obtain quadratic equation for R'

$$\frac{p_B(t) - p_\infty(t)}{\rho_L} = \frac{3}{2} R'^2 + \frac{4\nu_L}{R} R' \quad (10)$$

and its roots

$$R'_{1,2} = \frac{-\frac{4\nu_L}{R} \pm \sqrt{(4\nu_L / R)^2 + 6[p_B(t) - p_\infty(t)]/\rho_L}}{3}. \quad (11)$$

It is evident that the positive root gives less values of growth of the bubble, than according absolute value the negative root, which corresponds to fast decrease of bubble (with following collapse). In the first case, the bubble grows because of diffusion of vapour from the outside (naturally, under heating). In the second case (when the source of heating is switched off) there the condensation of vapour takes place in the bubble and the former diminishes at once.

Similar analysis may be provided on the basis of Euler-Poisson's equation for evolution of vapour-gas bubble.

2. Above-considered equations were connected with pure liquid and the bubbles contained only vapour of that liquid. In general case the bubbles contain some quantity of contaminant gas, whose partial pressure is p_{G_0} at some reference size, R_0, and temperature, T_∞, and, if there is no appreciable mass transfer of gas to or from the liquid, it follows that for polytropic process

$$p_B(t) = p_V(T_B) + p_{Go} \left(\frac{T_B}{T_{\infty\infty}} \right) \left(\frac{R_0}{R} \right)^{3k}. \quad (12)$$

Having substitute equation (12) into (1), we obtain the Rayleigh-Plesset equation in the following general form, [2]:

$$\frac{p_V(T_\infty)-p_\infty(t)}{\rho_L}+\frac{p_{Go}}{\rho_L}\left(\frac{R_0}{R}\right)^{3k}=RR''+\frac{3}{2}(R')^2+\frac{4\nu_L}{R}R'+\frac{2\sigma}{\rho_L R}, \qquad (13)$$

After transference the second term from the left side of this equation to the right side, investigate on extremum corresponding functional (3), having made successively the operations (5)-(8), we obtain (for k = 1, isothermal process):

$$F_R=R''-\frac{4\nu_L}{R^2}R'-\frac{2\sigma}{\rho_L}\frac{1}{R^2}+\frac{p_{G_0}}{\rho_L}\frac{R_0^3}{R^4}, F_{R'}=3R'+\frac{4\nu_L}{R}, \frac{d}{dt}F_{R'}=-\frac{4\nu_L}{R^2}, F_{R''}=R, \frac{d^2}{dt^2}F_{R''}=R''. \qquad (14)$$

After substituting of the equation (14 into (7), we obtain

$$R^2R''=-\frac{2\sigma}{\rho_L}+3\frac{G}{R^2}, \qquad (15)$$

where $G=\frac{p_{G_0}}{\rho_L}R_0^3$. In the absence of the gas contamination in the bubble, instead of equation (15) we have the equation for extremal of pure vapour integral (9). Joint solution of the equations (13) and (15) yields

$$\frac{p_V(T_\infty)-p_\infty(t)}{\rho_L}=\frac{3}{2}R'^2+\frac{4\nu_L}{R}R'+2\frac{G}{R^3}. \qquad (16)$$

It is evident, that when the bubble consists only the vapour of liquid, then G = 0, and the equation (16) coincides with the equation (10). At last, the integration of the equations (10) and (16) will allowed us to find the extremals of this variational task.

3. Among others, the stability to nonspherical disturbances has been investigated from a purely hydrodynamic point of view by Birkhoff (1954) and Plesset and Mitchell (1956), [1]. These analyses essentially examine the spherical equivalent of the Rayleigh-Taylor instability; they do not include thermal effects. If the inertia of the gas in the bubble is assumed to be negligible, then the amplitude, $a(t)$, of a spherical harmonic distortion of order *n (n > 1)* will be governed by the equation

$$\frac{d^2a}{dt^2}+\frac{3}{R}\frac{dR}{dt}\frac{da}{dt}-\left[\frac{(n-1)}{R}\frac{d^2R}{dt^2}-(n-1)(n+1)(n+2)\frac{\sigma}{\rho_L R^3}\right]a=0. \quad (17)$$

It is clear from (17) that the most unstable circumstances occur when $dR/dt < 0$ and $d^2R/dt^2 \geq 0$. These conditions will be met just prior to be rebound of a collapsing cavity. On the other hand, the most stable circumstances occur when $dR/dt > 0$ and $d^2R/dt^2 < 0$, which is the case for growing bubbles as they approach their maximum size.

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მეტასტაბილურ სითხეში აირორთქლოვანი ბუშტების ზრდაროგორც ვარიაციული ამოცანა

ა. გველესიანი

რეზიუმე

თეორიულად განიხილება ბუშტის ზრდის პროცესი შემთხვევაში, როდესაც ბუშტი შევსებულია მხოლოდ სითხის ორთქლით და აგრეთვე როდესაც ბუშტი წარმოადგენს ორთქლის და აირის ნარევს. პროცესი აღიწერება განზოგადებული რელეი-პლესეთის (რპ) განტოლებით, რომელიც ფუნქციონალის სახით შეისწავლება ვარიაციული აღრიცხვის მეთოდის გამოყენებით. ორივე შემთხვევისათვის მიღებულია ეილერ-პუასონის (ეპ) განტოლება, რომლის ექსტრემალების მოძებნა უფრო იოლია, ვიდრე (რპ)-ს ამოხსნა. დასასრულს ზოგად შემთხვევაში განიხილება სფერული ბუშტის რელეი-ტეილორის არამდგრადობა არასფერული შეშფოთებების დროს და, როგორც კერძო შემთხვევა, სფერული ბუშტის რადიალური შეშფოთება. განიხილება იზოთერმული. პროცესის შემთხვევა.

Рост газо-паровых пузырьков в метастабильных жидкостях, как вариационная задача

А.И. Гвелесиани

Резюме

Теоретически рассматривается процесс роста пузырька в случае, когда пузырь заполнен паром чистой жидкости и в случае, когда в пузыре наряду с паром имеется газовая примесь. Исследование ведётся на основе обобщённого уравнения Релея-Плессета вариационным методом скорейшего спуска. Для обоих случаев получены соответствующие уравнения, интегрирование которых позволит найти экстремали поставленной вариационной задачи. В заключение рассматривается проблема устойчивости сферического пузырька в общем случае неустойчивости Релея-Тейлора при несферических возмущениях.