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Assessment of the WKB method error by the Gratton kinematic model for the Earth's magnetic boundary layer task

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Abstract

The work touches the flat model of the magnetic boundary layer corresponding to the meridional section of the magnetosphere. It solves the single-component equation of the magnetic induction, which matches so-called Zhigulevs's first order the magnetic boundary layer, in which field of speed is given by the modified Gratton's kinematic model for the compressible solar wind. The work also describes the obtained exact numerical solution to the above mentioned equation and approximate analysis solution by the Wentzel-Kramers-Brillouin (WKB) method. The boundary conditions correspond to the area containing critical points at the dayside and night side of the magnetosphere, i.e. in the plasmasphere. It defines the error ($\approx 1\%$) of the WKB method and the linear size of the use area(1-0,6) of this method. It assesses the compressibility effect of the solar wind, which must be influencing on the topological image of the magnetic field distribution in the magnetopause and the area alongside the plasmapause.

During the interaction of the solar wind and the geomagnetic field a special structure is formed. It is the magnetopause, which by its features resembles the boundary layer. The mathematical modeling of this formation must be done on the basis of magneto-hydrodynamic (MHD) equations. Namely, as a result of their simplification so called Zhigulev's equations for the magnetosheath of I and II order are received. These equations appeared to correspond to the main sections of the Earth's magnetosphere: I order system corresponds to the meridional section of the magnetosphere, and II order system – to the equatorial one. The main problem for the magnetopause modeling is self-consistency of the solar wind flow and the geomagnetic field that appeared impossible for a general case. In order to prevent this problem we used different kinematic models of the velocity field. In the gasodynamic approximation these models give an image of the cosmic plasma flow near the critical point of the magnetosphere [1-3]. The most popular among these models appeared the Parker kinematic model and its modifications, by means of which the main parameters of the magnetopause were analytically obtained: thickness and profiles of the magnetic field [4]. The advantage of these solutions is the physical obviousness, though they have a significant lack: the defects of analytical solution are mainly caused by two factors:

I. *errors of the approximation analysis methods (e.g. the Shwets method);*
II. *shortage of measures in the use area of the Parker kinematic model (the focal area containing the critical point).*

Therefore the work [5] described the flat modification of so called Gratton kinematic model. This method obviously more matches the solar wind flow in the central area of the magnetosheath, which is in fact a maximum size focal area. Hereby, let us note that, apart the day side of the magnetosphere, the Gratton model is admissible for the night side (the magnetosphere tail) as well. Namely, according to the topology of the geomagnetic field force lines, there is a critical point at the night side of the plasmasphere. Consequently, there is a focal area, which is formed during convective motion directed from the neutral layer to the sun. Such special cases are recorded by scientific satellites during strong perturbations of the solar wind when reconnection of the geomagnetic field bounding the plasmic layer takes place in the magnetosphere tail [6].

The goal of the work is determination of topological image of the geomagnetic field distribution in the meridional section and assessment of the errors of the approximation analysis solution of the magnetic field induction equation in case we use Gratton's flat compressible kinematic model. This task has a practical value in the viewpoint of modeling of the magnetopause immediately as well as for the assessment of the meridional magnetopause parameter errors obtained by the second order magnetic boundary layer equation. This error is caused by the Parker kinematic model and the Shwets sequence approximation model, precision of which was assessed earlier in regard to several accurate solutions and is approximately 15-20% [7]. At the same time, below, within the framework of our task, we have assessed the error of the Wentzel-Kramers-Brillouin (WKB) method. It is known that this method is especially effective for the solution of second order differential equations with varying coefficients of the following type:

$$y'' + f(t)y' + g(t)y = 0. \qquad (1)$$

In the case of just some coefficients it is impossible to obtain an exact solution for (1) equation. This fact is a limitation for the physical task. Namely, we have such a situation in the case of meridional magnetopause modeling, which is quite observable if we use the Gratton model:

$$\begin{cases} V_x = -U_0\left(1 - e^{-\frac{U_0}{\nu}x}\right), \\ V_y = U_0 \cdot K_y \cdot y \cdot e^{-\frac{U_0}{\nu}x}, \end{cases} \qquad (2)$$

where U_0 is a velocity characteristic of the solar wind, ν - magnetic viscosity, K_y - a reverse value of the linear scale. Here a coordinate system with its origin in the critical point is used: X axis is either directed toward the sun (in the case of the day side of the magnetosphere) or opposite to the sun (at the night side of the magnetosphere); Y axis determines the direction of the extreme line of the geomagnetic field. In regard to Z axis the model is homogenous as the compressibility of the solar wind plasma is postulated. From the continuity equation we will receive

$$div\vec{V} = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} = -U_0 \cdot \frac{U_0}{\nu} e^{-\frac{U_0}{\nu}x} + U_0 K_y e^{-\frac{U_0}{\nu}x} = \theta t, \qquad (3)$$

where a notation $t = e^{-\frac{U_0}{\nu}x}$ is used.

If we disregard the curvature of the geomagnetic field it will be sufficient to look at the equation of the single-component magnetic field induction, which corresponds to the meridional magnetic boundary layer:

$$H_y \frac{\partial}{\partial y} V_y - V_x \frac{\partial}{\partial x} H_y - H_y \theta e^{-\frac{U_0}{\nu}x} = -\nu \frac{\partial^2 H_y}{\partial x^2}. \qquad (4)$$

It is quite obvious that (4) equation does not change during the variation in the magnetic field force line direction. This means that the equation really corresponds to the day side as well as the night side of the magnetosphere. If we refer to a new variable: $t = e^{-\frac{U_0}{\nu}x}$ and take into account that $\frac{dH_y}{dx} = \frac{dH_y}{dt}\frac{dt}{dx} = \left(-\frac{v_0}{v}\right)\frac{dH_y}{dt}t$, $\frac{d^2H_y}{dx^2} = \frac{v_0^2}{v^2}\left(\frac{d^2H_y}{dt^2}t^2 + \frac{dH_y}{dt}t\right)$ by simple transformations we will receive the equation of (1) equation type

$$\frac{d^2H_y}{dt^2} + \frac{dH_y}{dt} + \frac{1}{t}H_y = 0. \qquad (5)$$

The exact numerical solution to the equation requires boundary conditions, which are similar to each other in the cases of both the magnetopause and the plasmasphere of the magnetosphere night side

$$H_y = H_0 \;, \quad \text{when } t=1, \qquad \frac{dH_y}{dt} = 0. \qquad (6)$$

where H_0 is the value characteristic of the geomagnetic field at the lower boundary of the magnetopause or the night side of the plasmapause. The second criterion of (6) equation physically means that there is disregard of the surface magnetospheric DCF- current effect, which always exists in the magnetopause, whereas is absent in the plasmapause. This limitation is not significant for our task as it is always possible to indirectly take into consideration the magnetic effect of the DCF- current in the way of varying the value characteristic of the geomagnetic field.

As we have mentioned above (5) equation may be solved also by the WKB approximation analysis method. By the scheme of WKB the equation (5) gives the following equation:

$$V'' + h(t)V = 0 \,, \qquad (7)$$

V is connected with H_y, and $h(t)$ coefficient is determined by means of the coefficients of (5) equation [8]. Namely, in our case $h(t) = \frac{1}{t} - \frac{1}{4}$.

For the solution, according to the WKB method, let us have $V = e^{i\Phi(t)}$ notation. Consequently, we will receive a nonlinear equation as follows:

$$-(\Phi')^2 + i\Phi'' + h = 0 \,. \qquad (8)$$

In the first approximation, i.e. when the number with an imaginative coefficient is disregarded, from (8) equation we will receive

$$\Phi' = \pm\sqrt{h} \,, \quad \text{i.e.} \quad \Phi = \pm\int\sqrt{h}dt \;. \qquad (9)$$

It is natural that such a supposition is correct only in the case the following condition is fulfilled:

$$|\Phi''| \approx \frac{1}{2}\left|\frac{h'}{\sqrt{h}}\right| << h \,. \qquad (10)$$

If we use the expression $h(t) = \frac{1}{t} - \frac{1}{4}$, it will be quite obvious that condition $\left(\frac{1}{2}\left|\frac{h'}{\sqrt{h}}\right| \le h\right)$ is roughly satisfied only in this interval $t \in (1 - 0{,}6)$, and not in the whole interval

(1-0). The approximation next to Φ is searched by the iteration, for which in (8) equation let us suppose that $\Phi'' \approx \pm h^{-\frac{1}{2}} \cdot \frac{h'}{2}$. In this case we will have

$$(\Phi')^2 \approx h \pm \frac{i}{2}\frac{h'}{\sqrt{h}}, \qquad (11),$$

from which
$$\Phi' \approx \pm\sqrt{h} + \frac{i}{4}\frac{h'}{h}, \qquad (12),$$

i.e. there is
$$\Phi(t) \approx \pm\int \sqrt{h(t)}dt + \frac{i}{4}\ell n h(t), \qquad (13),$$

from which
$$V \approx \frac{1}{h^{1/4}}\left\{C_+ e^{i\int\sqrt{h}dt} + C\_ e^{-i\int\sqrt{h}dt}\right\}. \qquad (14)$$

If we correspond (5) equation to (1) general equation we receive that $f = 1$. Consequently, according to the WKB method $H_y = V(t)e^{-\frac{t}{2}}$. Thus, for the magnetic field we have the following general expression

$$H_y \approx \frac{e^{-\frac{t}{2}}}{\left(\frac{k}{t}-\frac{1}{4}\right)^{1/4}} H_0 \left\{C_+ e^{i\int\sqrt{\left(\frac{k}{t}-\frac{1}{4}\right)}dt} + C\_ e^{-i\int\sqrt{\left(\frac{k}{t}-\frac{1}{4}\right)}dt}\right\}. \qquad (15)$$

The constants C_+ and $C\_$ are determined by two algebraic equations that are received by the boundary conditions (6). Finally, we have

$$C_+ \approx -3.4086 + 0.2896i,$$
$$C\_ \approx -3.4086 - 0.2896i. \qquad (16)$$

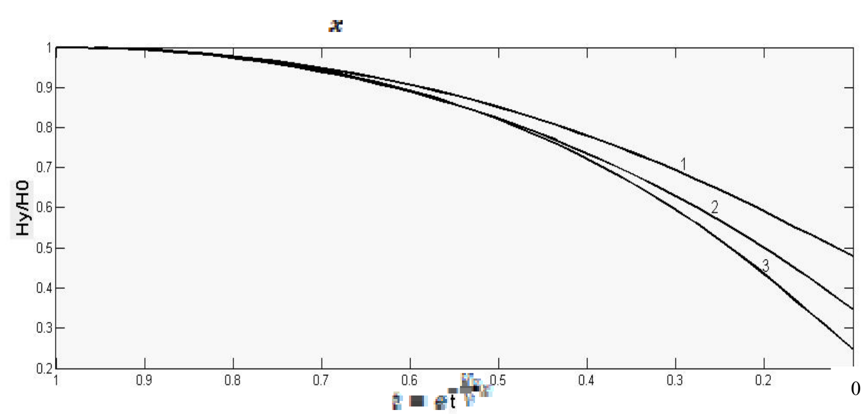

Fig.1

Fig.1 shows the algebraic normalized solution of (5) equation (line.1), the curve corresponding to (15) expression (line.2) and the numerical solution to the same equation in the case of non-compressible medium (θ=0) (line.3). It is obvious that between the exact numerical solution and the approximate analysis solution there is quite a good consistency in the some interval of $t \in (1 - 0{,}6)$ $(x \in (0, \infty))$. This means that if we refer to the initial coordinate system the exact and approximate solutions give in fact identical results of the magnetic field distribution near the critical point. The difference between (1) and (2) these solutions becomes significant after the point, from which the criterion (10) is not fulfilled $t = 0{,}6$. Its position in space is determined by parameter $\frac{u_0}{\nu}$, the value of which depends on the quality of the solar wind perturbation.

Conclusion

Thus, we may conclude that within the framework of our task the error of the WKB method does not exceed 1%, which is quite acceptable for the approximate analysis method. At the same time, it is noteworthy that according to our model the compressibility effect must by quite significantly influencing on the topological image of the magnetic field distribution.

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ვკბ მეთოდის ცდომილების შეფასება გრატონის კინემატიკური მოდელის გამოყენებით დედამიწის მაგნიტური სასაზღვრო ფენის ამოცანისათვის

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რეზიუმე

განხილულია დედამიწის მაგნიტური სასაზღვრო ფენის ბრტყელი მოდელი, რომელიც შეესაბამება მაგნიტოსფეროს მერიდიონალურ კვეთას. ამოხსნილია ე.წ. ჟიგულევის პირველი გვარის მაგნიტური სასაზღვრო ფენის შესაბამისი მაგნიტური ინდუქციის ერთკომპონენტიანი განტოლება, რომელშიც სიჩქარეთა ველი მოცემულია კუმშვადი მზის ქარისათვის მოდიფიცირებული გრატონის კინემატიკური მოდელით. მიღებულია ამ განტოლების ზუსტი რიცხვითი ამონახსნი და მიახლოებითი ანალიზური ამონახსნი ვკბ მეთოდით. სასაზღვრო პირობები შეესაბამება კრიტიკული წერტილების შემცველ არეს მაგნიტოსფეროს დღის მხარეზე და ღამის მხარეზე, ანუ პლაზმოსფეროზე. განსაზღვრულია ვკბ მეთოდის ცდომილება (≈ 1%) და ამ მეთოდის გამოყენების არის ხაზოვანი ზომა. შეფასებულია მზის ქარის კუმშვადობის ეფექტი, რომელიც მაგნიტოპაუზაზე და პლაზმოპაუზის მიმდებარე არეში საკმაო გავლენას უნდა ახდენდეს მაგნიტური ველის განაწილების ტოპოლოგიურ სურათზე.

Оценка ошибки метода WKB с помощью кинематическиой модели Гратона для задачи магнитного пограничного слоя Земли

М. Чхитунидзе, И. Хведелидзе, Н. Джонджоладзе

Резюме

В данной работе рассмотрена плоская модель магнитного пограничного слоя Земли, которая соответствует меридиональному сечению магнитосферы. Решено однокомпонентное уравнение магнитной индукции, соответствующее уравнению I первого рода Жигулева магнитного пограничного слоя Земли, в котором поле скоростей дано модиффицированной моделью Гратона для сжимаемого солнечного ветра. Получено точное численное решение этого уравнения и приближённое аналитическое решение методом Вентцел-Крамер-Брилвена. Граничные условия соответствуют области, содержащей критической точке на дневной и ночной сторонах магнитосферы, то есть на плазмосфере. Определены ошибка метода ВКБ (≈ 1 %) и линейный размер области использования этого метода. Оценен эффект сжимаемости солнечного ветра, который существено должен влиять на топологическую кортину распределения магнитного поля на магнитопаузе и в области плазмопаузы.