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Confidence Interval of Parameters for Gaussian Statistical Structures Z-Criteria's Application

[1]Zurab S. Zerakidze , [2]Jemal K. Kiria, [2]Tengiz V. Kiria

[1]*Gori State University, Gori, Georgia*
[2]*M. Nodia Institute of Geophysics of I. Javakhishvili Tbilisi State University, Georgia*

ABSTRACT

In this paper is proven 100% confidence interval of parameters for Gaussian statistical structures in Banach space of measures.
Key words: *Gaussian statistical structure, consistent estimators of parameters, Z-criteria, orthogonal structure, strongly separable structure, confidence interval of parameters.*

Introduction

Recall that a statistical criterion is any measurable mapping from the set all possible samples values to the set of hypothesis. It is said that an error of h-th kind of the δ criterion occurs, if the criterion ejects the main hypothesis of H_h. The following $\alpha_h(\delta) = \mu_h(\{x: \delta(x) \neq h\})$ is called the probability of an error of the h-th kind for a given criterion δ.

The notion and corresponding construction of Z-criteria (same "Generalization criterion of Neiman-Pearson, consistent criterion") for hypothesis testing were introduced and studied by Z. Zerakidze (see [2-13]).

We recall some definitions from the works [1-14].

Let (E, S) be a measurable space. The density of Gaussian law is determined by the equality

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-a)^2}{2\sigma^2}}$$

Let μ be the probability measure given on $([-\infty, +\infty), L[-\infty, +\infty))$ by the formula $\mu(A) = \int_A f(x)dx, A \in L[-\infty; +\infty)$, where $L([-\infty, +\infty))$ is Lebesgue σ-algebra. Let $\{\mu, i \in I\}$ be Gaussian measures.

Definition 1. An object $\{E, S, \mu, i \in I\}$ is called an Gaussian statistical structure. Definition 2. An Gaussian statistical structure $\{E, S, \mu_,, i \in I\}$ is called orthogonal if μ_i and μ_j are orthogonal for each $\forall i \neq j, i \in I, j \in I$.

Definition 3. An Gaussian statistical structure $\{E, S, \mu, i \in I\}$ is called weakly separable if there exists a family of S-measurable sets $\{X, i \in I\}$ such that the relations are fulfilled:

$$(\forall i)(\forall j)(i \in I \& j \in I) \Rightarrow \mu_i(X_,) = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases}$$

Let $\{\mu_{,}i \in I\}$ be Gaussian measures defined on the measurable space (E,S). For each $i \in I$ we denote by $\bar{\mu}_i$ the completion of the measure μ_i, and by $\text{dom}\,(\bar{\mu}_i) -$ the $\sigma -$ algebra of all $\mu_i -$ measurable subsets of E . We denote $S_1 = \bigcap_{i\in I} \text{dom}\,(\bar{\mu}_i)$.

Definition 4. The Gaussian statistical structure $\{E, S, \mu_I, i \in I\}$ is called strongly separable Gaussian statistical structure if there exists a family of S-measurable sets $\{Z_i, i \in I\}$ such that the relations are fulfilled

1 $\mu_i(Z_i) = 1, \forall i \in I;$

2 $Z_i \cap Z_j = \emptyset \forall i \neq j; i, j \in I$

3 $\bigcup_{i\in I} Z_i = E.$

Let I be set of parameters and B(I) be σ-algebra if subsets of I which contains all finite subsets of I.

Definition 5 . We will say that the Gaussian statistical structure $\{E, S_1, \bar{\mu}_i, i \in I\}$ admits a consistent estimators of parameters if there exists at least one measurable mapping $f: (E, S_1) \to (I, B(I))$, such that $\bar{\mu}_i(\{x: f(x) = i\}) = 1, \forall i \in I.$

Let H be set of hypotheses and $B(H)$ be σ-algebra of subsets of H which contains alle finite subsets of H .

Definition 6. We will say that the Gaussian statistical structure $\{E, S_1, \bar{\mu}_h, h \in H\}$ admits Z-criterion (same "Generalization Neimana-Pearson,consistent criterion") for hypothesis testing if there exists at least one measurable mapping $\delta: (E, S_1) \to (H, B(H))$, such that

$\bar{\mu}_h(\{x: \delta(x) = h\}) = 1, \forall h \in H.$

Definition 7. The probability $\alpha_h(\delta) = \bar{\mu}_h(\{x: \delta(x) \neq h\})$ is called the probability of error of hth kind for the given criterion δ.

Theorem 1. The Gaussian statistical structure $\{E, S_1, \bar{\mu}_h h \in H\}$ admits a Z-criterion (same "Generalization Neimana-Pearson, consistent criterion") for hypothesis testing if and only if this probability of error of kind is equal to zero for the criterion δ.

Proof. Necessity. Since the statistical structure $\{E, S_1, \bar{\mu}_h h \in H\}$ admits a Z-criterion countable Gaussian statistical structure $\{E, S, \mu_h, h \in H\}$ admits a Z-criterion for hypothesis testing, there exists a measurable mapping $\delta: (E, S_1) \to (H, B(H))$, such that $\bar{\mu}_h(\{x: \delta(x) = h\}) = 1, \forall h \in H$. Therefore, $\alpha_h(\delta) = \bar{\mu}_h(\{x: \delta(x) \neq h\}) = 0, \forall h \in H.$

Sufficiency. Since the probability of any kind is equal to zero, have $\alpha_h(\delta) = \bar{\mu}_h(\{x: \delta(x) \neq h\}) = 0, \forall h \in H.$

On other hand, $\mu\{x: [(\delta(x) = h) \cup (\delta(x) \neq h)]\} = \bar{\mu}_h(\{x: \delta(x) = h\}) = 1, \forall h \in H.$

2. Confidence interval for of parameters Gaussian statistical structures in Banach space of measures

Let M^σ be a real linear space of all alternating finite measures on S.

Definition 8. A linear subset $M_B \subset M^\sigma$ is called a Banach space of measures if:

1 The norm on M_B can be defined so that M_B it is Banach space with respect to this norm, and the inequality $\| \mu + \lambda v \| \geq \| \mu \|$ holds for any orthogonal measures $\mu, v \in M_B$ and real number $\lambda \neq 0$;

2 If $\mu \in M_B$ and $|f(x)| \leq 1$, then $v_f(A) = \int_A f(x)\mu(dx) \in B_B$ and $\|v_f\| \leq \| \mu \|$;

3 If $v_n \in M_B, v_n > 0, v_n(E) < \infty, n = 1,2, \dots$ and $v_n \downarrow 0$, then for any linear functional $l^* \in M_B^*$: $\lim_{n\to\infty} l^*(v_n) = 0$, where M_B^* conjugate to linear space M_B.

Remark 1. The definition and construction of a Banach space of measures were given by Z . Zerakidze (see [14]).

Definition 8. Let I be a set of indexes and M_{B_i} is a Banach space for all $i \in I$. The Banach space $M_B = \{X_i\}_{i\in I}: X_i \in M_{B_i}, \forall i \in I, \sum_{i\in I} \|X_i\| \leq 0\}$ with the norm $\|X_i\|_{i\in I} = \sum_{i\in I} \|X_i\|_{M_{B_i}}$ is called the direct sum of Banach space M_{B_i} and is denoted by $M_B = \bigoplus M_{B_i}$.

Remark 2. Obviously, any Banach space of measures is a Banach space the elements of which are alternating measures, but not vice versa. The following theorem was proved in [14].

Theorem 2. Let M_B be a Banach space of measures, then there exists the funnily of pairwise orthogonal probability measures $\{\mu_{h_i}, i \in I\}$, Card $I = 2^{2^c}$, such that $M_B = \bigoplus M_{B_i}(\mu_{h_i})$ is Banach space of elements v of the from

$v(B) = \int f(x)\mu_{h_i}(dx), B \in S, \int |f(x)|\mu_{h_i}(dx) < +\infty$, with the norm

$\| v \|_{M_{B_i}(\mu_{B_i})} = \int |f(x)|\mu_{h_i}(dx)$.

We define by $\mathrm{F} = \mathrm{F}(M_B)$ the set of real function f such that $\int f(x)\bar{\mu}_h dx$ is defined all $\bar{\mu}_h = M_B$. Theorem 3. Let $M_B = \bigoplus M_{B_i}(\bar{\mu}_h)$, Card $H \leq c$ be the Banach space of measures, E be a complete separable metric space, $S_1 = \bigcap_{h\in H} \mathrm{dom}\,(\bar{\mu}_h)$ is a Borel σ-algebra on E. In order for the Borel orthogonal Gaussian statistical structure $\{E, S_1, \bar{\mu}_h h \in H\}$, Card $H = c$ to admit Z-criterion (same "Generalization Neimana; Pearson consistent criterion") for hypothesis testing in the theory (ZFC)&(MA) it is necessary and sufficient the correspondence $f \leftrightarrow h_f$ defined by the

equality $\int f(x)\bar{\mu}_h(dx) = l_f(\bar{\mu}_h)$, $\bar{\mu}_h \in M_B$ was one-to-one (here l_f is a linear continuous functional on $M_B, f \in F(M_B)$.

Proof. Necessity. The existence of Z-criterion for hypothesis testing $\delta: (E, S_1) \to (H, B(H))$, implies that $\bar{\mu}_h(\{x: \delta(x) = h\}) = 1, \forall h \in H$. Setting $X_h = (\{x: \delta(x) = h\}) = 1, \forall h \in H$ we get:

1 $\bar{\mu}_h(X_h) = 1, \forall h \in H$;

2 $X_{h'} \cap X_{h''} = \emptyset$ for all different h' and h'' from H;

3 $\bigcup_{n\in H} X_h = \{x: \delta(x) \in H\} = E$.

Therefore the Gaussian statistical structure $\{E, S_1, \bar{\mu}_h h \in H\}$ is strongly separable, hence, there exists S_1 - measurable sets $\{X_h, h \in H\}$ such that $\bar{\mu}_h(X_{h'}) = \begin{cases} 1, & \text{if } h = h' \\ 0, & \text{if } h \neq h' \end{cases}$

We put the linear continuous functional l_{c_h} into correspondence to function by the formula $\int l_{c_h}(x)\bar{\mu}_h(dx) = l_{c_h}(\bar{\mu}_h) = \|\bar{\mu}_h\|_{M_B(\bar{\mu}_h)}$.

Let l_{X_h} be a linear continuous functional that correspondence to the function $\bar{f_1}(x) = f_1(x)I_{X_h}(x)$. Then for any $\bar{\mu}_{h_1} \in M_B(\bar{\mu}_h)$ we have

$$\int \bar{f_1}(x)\bar{\mu}_{h_1}(dx) = \int f_1(x)f(x)I_{X_h}(x)\bar{\mu}_h(dx) = l_{\bar{f_1}}(\bar{\mu}_{h_1}) = \|\bar{\mu}_{h_1}\|_{M_B(\bar{\mu}_h)}.$$

Let $\sum$ be the set of extensions of a functional that satisfy the condition $l_f \leq p(x)$ in those subspace where they are defined. Lets introduce a partial ordering into, assuming $l_{f_1} < l_{f_1}$ if f_2 is defined on a large set than l_{f_1} and $l_{f_1} = l_{f_2}$ where both of them are defined.

Let $\{l_{f_h}\}_{h\in H}$ be a linear ordered subsid in $\sum, M_B(\bar{\mu}_h)$ the subspace on which l_{f_h} is defined. We define $l_f \in\cup M_B(\bar{\mu}_h)$ setting $l_f(\mu) = l_{f_h}(\mu)$ if $\mu \in M_B(\bar{\mu}_h)$. It is obvious that $l_{f_h} < l_f$. Since any lineally ordered subset in $\sum$ has an upper bound due to the Chorn lemma $\sum$ contains the maximal element λ defined on some set X' satisfying the condition $\lambda \leq \mathrm{p(x)}$ for $x \in X'$. But X' must coincide with the entire space M_B because otherwise we could extended λ to a wider space by adding as above one more dimension. This contradicts the maximality of λ and, hence $X' = M_B$. Therefore, the extension of the functional is defined everywhere.

Let l_f be a linear functional that corresponds to the function $f(x) = \sum g_h(x)I_{X_h}(x) \in F(M_B)$. Then we have $\int f(x)\mu(dx) = \| \mu \| = \sum \|\bar{\mu}_h\|_{M_B(\bar{\mu}_h)}, M_B(\bar{\mu}_h)$ where

$$\mu(B) = \sum \int g_h(x)\bar{\mu}_h(dx),\ B \in S_1$$

Sufficiency. If for each $f \in F(M_B)$ the integral $\int f(x)\bar{\mu}_h(dx), \forall\bar{\mu}_h \in M_B$, is defined them there exist a countable subsets I_f in H for which $\int f(x)\bar{\mu}_h(dx) = 0$, if $h\bar{\epsilon}I_f$, $\sum\int |f(x)|\bar{\mu}_h(dx) < \infty$ and for any countable subset $\bar{I} \subset H$ and for the measure

$v(c) = \int_{h\in i}\int_c g_h(x)\bar{\mu}_h(dx)$ we have $\int_E f(x)v(dx) = \sum_{h\in I_f\cap\bar{I}}\int_E f(x)g_h(x)\bar{\mu}_h(dx)$.

Let the correspondence $f \to l_f$ be calefied the equality $\int_E f(x)\bar{\mu}_h(dx) = l_f(\bar{\mu}_h)$, then for $\bar{\mu}_{h_1}$, $\bar{\mu}_{h_2} \in M_B(\bar{\mu}_h)$ we have $\int_E f_{h_1}(x)\bar{\mu}_{h_2}(dx) = l_{f_{h_1}}(\bar{\mu}_{h_2}) = \int_E f_1(x)f_2(x)\bar{\mu}_{h_1}(dx) = \int_E f_{h_1}(x)f_{h_2}(x)\bar{\mu}_{h_1}(dx)$.

Therefore $f_{h_1}(x) = f_1(x)$ almost everywhere with respect to the measure $\bar{\mu}_{h_1}$. Let $f_{\bar{\mu}_{h_1}}(x) > 0$ almost everywhere with respect to $\bar{\mu}_{h_i}$ and $\int_E f_{\bar{\mu}_h}(x)\bar{\mu}_h(dx) < \infty$. If we denoyte now $\bar{\mu}_h(c) = \int_c f_{\bar{\mu}_h}(\mathrm{x})\bar{\mu}_h(dx)$, the we obtain $\int_E f_{\bar{\mu}_h}(x)\bar{\mu}_{h'}(dx) = l_{f_{\bar{\mu}_h}}(\bar{\mu}_{h'}) = 0, \forall h \neq h'\ \forall\bar{\mu}_h \in M_B(\bar{\mu}_h)$.

Denote by $C_h = \{x: f_{\bar{\mu}_h}(x) > 0\}$. Then $\mu_{h'}(C_h) = 0 \forall h \neq h'$. Therefore, there exist $S_1 -$ measurable sets $(h \in H)$ such that that $\mu_h(X_{h'}) = \begin{cases} 1, & \text{if } h = h' \\ 0, & \text{if } h \neq h' \end{cases}$ and hence the Gaussian statistical structure

$\{E, S_1, \bar{\mu}_h h \in H, \text{cardH} = c\}$ is weakly separable. We represent as an inductive sequence $\{\bar{\mu}_h < w_1\}$ where w_1 denotes the first ordinal number of the power of the set H .

We define w_1 sequence Z_h of parts of the E such that the following relations hold: 1) Z_i is Borel subset of E, $\forall h < w_1$; 2) $Z_h \subset X_h, \forall h < w_1$; 3) $Z_h \cap Z_{h'} = \emptyset$ for all $h' < w_1, h = h'$; 4) $\bar{\mu}_h(Z_h) = 1, \forall h < w_1$.

Suppose that $Z_{h_0} = X_{h_0'}$. Suppose that the partial sequence $\{Z_{h'}\}_{h'<h}$ is already defined for $h < w_1$. It is clear that $\mu^*(\cup_{h'<h} Z_{h'}) = 0$. Thus there exists a Borel subset y_h of the space E such that the following relations are valid $\cup_{h'<n} y_h$ and $\mu^*(y_h) = 0$. Assuming that $Z_h = X_h\ y_h$, we construct the w_1 sequence $\{Z_h\}_{h<w_1}$ of disjunctive measurable subsets of the space E. Therefore, $\mu_h(Z_h) = 1, \forall h < w_1$ and the Gaussian statistical structure $\{E, S_1, \bar{\mu}_h h \in H, \text{cardH} = c\}$ is strongly separable because that exists a family of elements of the σ-algebra $S_1 = \cap_{h \in H} \text{dom}\,(\bar{\mu}_h)$ such that 1) $\bar{\mu}_h(Z_h) = 1, \forall h \in H$; 2) $Z_{h'} \cap Z_h = \emptyset, \forall h' \neq h$; 3) $\cup_{h \in H} Z_h = \text{E}$.

For $\text{x} \in E$, we put $\delta(x) = h$, where h is the unique hypothesis from the set H for which $\text{x} \in Z_h$. The existence of such a unique hipotez from H can be proved using conditions 2), 3).

Let now $y \in B(H)$. Then $\{x: \delta(x) \in y\} = U_{h \in H} Z_h$.

It $h_0 \in H$, then $\{x: \delta(x) \in y\} = U_{h \in H} Z_h = Z_{h_0} \cup (U_{h \in H} Z_h)$. On the other hand the validity of the condition $\cup_{h \in H} Z_h \subseteq E - Z_{h_0}$ implies that $\bar{\mu}_{h_0}\left(\cup_{h \in y - h_0} Z_h\right) = 0$. The last equality yields $\cup_{h \in y - h_0} Z_h \in \text{dom}\,(\bar{\mu}_{h_0})$. Since $\text{dom}\,(\bar{\mu}_{h_0})$ is a σ-algebra, we deduce that $\{x: \delta(x) \in y\} \in \text{dom}\,(\bar{\mu}_h)$.

If $h_0 \notin y$, then $\{x: \delta(x) \in y\} = U_{h \in H} Z_h \leq \left(E - Z_{h_0}\right)$ and we conclude that $\bar{\mu}_{h_0}(\{x: \delta(x) \in y\}) = 0$. The last relation implies that $\{x: \delta(x) \in y\} \in \text{dom}\,(\bar{\mu}_{h_0})$.

We have shown that the map $\delta: (E, S_1) \to (H, B(H)$ is a measurable map. Since B(H) contains all singletons of H we as certain that $\bar{\mu}_h(\{x: \delta(x) = h\}) = \bar{\mu}_h(Z_h) = 1, \forall h \in H$.

The following Theorem is proven to Theorem 2.

Theorem 3. Let $M_B = \oplus M_B(\bar{\mu}_i)$, Card $I \leq c$ be the Banach space of measures, E be a complete metric space, $S_1 = \bigcap_{i \in I} \text{dom}\,(\bar{\mu}_i)$ is a Borel σ-algebra on E. In order for the Borel orthogonal Gaussian statistical structure $\{E, S_1, \bar{\mu}_i, i \in I\}$, Card $I \leq c$ to admit consistent estimator of parameters it is necessary and sufficient that correspondence $f \leftrightarrow l_f$ defined by the equality $\int f(x) \bar{\mu}_i(dx) = l_f(\bar{\mu}_i), \bar{\mu}_i \in M_B$ was one-to-one (have l_f is a linear continuous functional on $M_B, f \in F(M_B)$.

The following Theorems 1,2,3,4 follows that exponentials Gaussian structures existence consistent estimator of parameters Z-criterion for hypothesis testing and 100% confidence interval of parameters.

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პარამეტრების ნდობის ინტერვალი გაუსის სტატისტიკური სტრუქტურებისათვის Z - კრიტერიუმის გამოყენებით

ზ. ზერაკიძე, ჯ. ქირია, თ. ქირია

რეზიუმე

ნაშრომში განმარტებულია გაუსის ორთოგონალური, სუსტად განცალებადი, განცალებადი და ძლიერად განცალებადი სტატისტიკური სტრუქტურები. ასევე განმარტებულია პარამეტრების ძალდებული შეფასებები და პარამეტრების ჰიპოთეზათა შემოწმების სტრუქტურებისათვის Z - კრიტერიუმი (იგივეა, რაც „განზოგადოებული ნეიმან-პირსონის კრიტერიუმი“, „ძალდებული კრიტერიუმი“).

აგებულია გაუსის ალბათობების ზომების მიხედვით ბანახის ზომათა სივრცე და დამტკიცებულია ამ სივრცეში აუცილებელი და საკმარისი პირობები პარამეტრების ძალდებული შეფასებების და Z - კრიტერიუმის არსებობის შესახებ.

აგებულია გაუსის სტატისტიკური სტრუქტურების პარამეტრებისათვის 100%-იანი ნდობის ინტერვალი.
საკვანძო სიტყვები: გაუსის სტატისტიკური სტრუქტურა, თანმიმდევრული პარამეტრების შეფასებები, Z-ტესტი, ორთოგონალური სტრუქტურა, მკაცრად განცალკევებული სტრუქტურა, პარამეტრების ნდობის ინტერვალი.

Доверительный интервал параметров для статистических структур Гаусса с использованием Z-критерия

З. Зеракидзе, Дж. Кирия, Т. Кирия

Аннотация

В статье объясняются ортогональные, слабо разделимые, разделимые и сильно разделимые статистические структуры Гаусса. Также даются пояснения о вынужденных оценках параметров и Z-критерии для проверки гипотез о параметрах статистических структур (аналогичен «обобщённому критерию Неймана-Пирсона», «вынужденному критерию»). На основе вероятностных мер Гаусса построено пространство размерностей выборки и доказаны необходимые и достаточные условия существования вынужденных оценок параметров и Z-критерия в этом пространстве.
Для параметров статистических структур Гаусса построен 100%-й доверительный интервал.
Ключевые слова: гауссовская статистическая структура, состоятельные оценки параметров, Z-критерий, ортогональная структура, сильно разделимая структура, доверительный интервал параметров.